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The Bipolar Hydrodynamic Model for Semiconductors and the Drift–Diffusion Equations

R. Natalini

Istituto per le Applicazioni del Calcolo "M. Picone," Viale del Policlinico 137, I-00161 Rome, Italy

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We establish the existence of entropy solutions for a bipolar hydrodynamic model for semiconductors. We show also that the limit, as the relaxation time goes to zero, of an appropriate (scaled) sequence of entropy solutions is a solution of the classical drift-diffusion equations. © 1996 Academic Press, Inc.

1. INTRODUCTION

In recent years the theory of semiconductor device modeling has become an area of increasing interest in applied mathematics. For the moment to deal with the basic kinetic transport equations remains too expensive for real life applications. Nevertheless from transport equations it is possible to derive simpler fluid dynamical equations for macroscopic (integral) quantities like particle, current, or energy densities. They represent a good compromise between physical accuracy and the reduction of computational cost.

A standard approach for this derivation is the moments method. According to the different ansatz for the phase space densities, introduced to prescribe the dependence on the velocity, we recover different limit models and, in particular, the drift–diffusion equations and the hydrodynamic Euler–Poisson system (see [28, 6, 5], the reference book of Markovich, Ringhofer, and Schmeiser [24], and the literature quoted therein). More general hydrodynamic models have been derived in [2, 3, 4].

The bipolar drift–diffusion model was first derived in [28] and is the most popular model for simulations in semiconductor devices. This model works very well under the assumptions of low carrier densities and small electric fields. By contrast, the hydrodynamic models are usually considered to describe high field phenomena or submicronic devices.

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In this paper we first study a (scaled) bipolar hydrodynamic isentropic Euler–Poisson system for semiconductors. Namely, we consider the 1D bipolar Euler system

$$\begin{aligned} &\partial_t n + \partial_x j_n = 0, \\ &\partial_t p + \partial_x j_p = 0, \\ &\partial_t j_n + \partial_x\left(\frac{j_n^2}{n} + r_n\right) = nE + C_n, \\ &\partial_t j_p + \partial_x\left(\frac{j_p^2}{p} + r_p\right) = -pE + C_p, \end{aligned} \tag{BE}$$

supplemented by the classical Poisson equation

$$\partial_x E = n - p - b(x) \tag{P}$$

for all $(x,t) \in \mathbb{R} \times [0,\infty)$, $\tau > 0$; here n is the electron density, p the positively charged hole density, j_n and j_p are respectively the electron and the hole current densities, E stands for the (negative) electric field, and $b \in L^1$ for the density of the fixed (positively charged) background ions. The pressure–density relation is given for the electron pressure by $r_n(n) = (1/\gamma_n)n^{\gamma_n}$ and for the hole pressure by $r_p(p) = (1/\gamma_p)p^{\gamma_p}$, with $1 < \gamma_n, \gamma_p \leq 5/3$. The notations $u_n := j_n/n$, $u_p := j_p/p$ for the electron and hole velocities will be also used. The equations (BE) form a hyperbolic system of balance laws describing the motion of a gas of electrons and holes in semiconductors under the action of the electric field E. Then, by the general theory of the quasilinear hyperbolic equations [18, 26], the solutions can develop discontinuous states (shock waves) and need to be considered just in a weak sense.

The electron and the hole current relaxation terms C_n, C_p model the effects of collisions of electrons and holes with the semiconductor crystal lattice. Throughout this paper a simple relaxation time model is used for the terms C_n, C_p; i.e.,

$$C_n = -\frac{j_n}{\tau}, \qquad C_p = -\frac{j_p}{\tau},$$

where the constant $\tau > 0$ stands for the current relaxation time.

We are also concerned with the alternative model given by the following version of the drift–diffusion equations [23],

$$\begin{aligned} &\partial_t \mathcal{N} - \partial_{xx}\big(r_n(\mathcal{N})\big) + \partial_x(\mathcal{N}\mathcal{E}) = 0, \\ &\partial_t \mathcal{P} - \partial_{xx}\big(r_p(\mathcal{P})\big) - \partial_x(\mathcal{P}\mathcal{E}) = 0, \end{aligned} \tag{DD}$$

where the (negative) electric field $\mathscr{E}$ is still given by the Poisson equation (P), i.e.,

$$\partial_x \mathscr{E} = \mathscr{N} - \mathscr{P} - b.$$

Again $\mathscr{N}, \mathscr{P}$ stand for the concentrations of electrons and holes. The densities of the electron and the hole current are then recovered by the usual relations

$$\mathscr{I}_n = \mathscr{N}\mathscr{E} - \partial_x r_n(\mathscr{N}), \qquad \mathscr{I}_p = -\mathscr{P}\mathscr{E} - \partial_x r_p(\mathscr{P}).$$

Observe that Eq. (DD) form a weakly coupled degenerate parabolic system and the solutions could be nonregular in the regions $\{\mathscr{N} = 0\}$ or $\{\mathscr{P} = 0\}$; see [12, 7] and the references therein.

The weak solutions to the hydrodynamic model in the unipolar case have been investigated by many authors, in particular, [1, 11] for the stationary case and [21] in the general (isentropic) case. Related results have been obtained in [29, 13, 25]. In [22], the authors studied the relaxation limit of the hydrodynamic model to the drift–diffusion equations. The problem (DD)-(P), in the bipolar case, was considered in [14–16], where the existence and the uniqueness of weak solutions is established on a bounded domain with suitable boundary conditions.

In the present paper we first obtain, in Section 3, the existence of weak entropy solutions for the system (BE)-(P), with uniform L^∞ and L^2 bounds with respect to the relaxation time τ. These solutions are the limit of a numerical approximation based on a modified version of the fractional step Lax–Friedrichs scheme.

In Section 4 we show the convergence of a sequence of (scaled) entropy solutions to (BE)-(P) and then we establish the existence of weak solutions to the Cauchy problem for (DD)-(P).

2. THE APPROXIMATING SCHEME

First we define the notion of weak (entropy) solution to the Cauchy problem for the system (BE)-(P), supplemented by the initial conditions

$$\begin{aligned} n(x,0) &= n^0(x), \qquad p(x,0) = p^0(x) \\ j_n(x,0) &= j_n^0(x), \qquad j_p(x,0) = j_p^0(x) \end{aligned} \tag{2.1}$$

$$E(x,0) = E^0(x) = \int_{-\infty}^{x} \left(n^0(y) - p^0(y) - b(y)\right) dy + E^-,$$

where n^0, p^0, j_n^0, j_p^0 are some given bounded measurable functions with compact support, $n^0 \geq 0$, $p^0 \geq 0$, and E^- is a constant. Set

$$V = (n, p, j_n, j_p),$$

$$F(V) = \left(j_n, j_p, \frac{j_n^2}{n} + r_n(n), \frac{j_p^2}{p} + r_p(p) \right),$$

$$G(V, E) = \left(0, 0, nE - \frac{j_n}{\tau}, -pE - \frac{j_p}{\tau} \right).$$

Then the system (BE) reads now

$$\partial_t V + \partial_x F(V) = G(V, E). \tag{2.2}$$

Let us recall some basic facts from the theory of the weak solutions to the quasilinear hyperbolic systems [18, 19, 26]. As is well known, the linear structure of the problem can generate discontinuous solutions even for very regular initial data and then we have to deal with weak solutions. It turns out that in general these weak solutions are not unique and some additional conditions (*entropy conditions*) have to be imposed to recover uniqueness. An entropy–entropy flux pair for a system of balance laws is a couple $(\eta, q) = (\eta(V), q(V))$, defined by the relation

$$\nabla q = \nabla \eta \, \nabla F,$$

where ∇ denotes gradient with respect to the state variable V.

We say that the vector-valued function (V, E) is an entropy solution of the system (BE)-(P) if the Poisson equation is taken in the weak (distributional) sense and for any convex entropy function $\eta = \eta(V)$ one has

$$\partial_t \eta + \partial_x q + S \leq 0 \qquad \text{in } \mathscr{D}', \tag{2.3}$$

for any locally bounded measurable function $S(x, t)$, such that

$$S + (\nabla \eta)^{\mathrm{T}} \cdot G(V, E) \leq 0.$$

The characteristic velocities for the system (BE) are

$$\begin{aligned} \lambda_{n_-} &= \frac{j_n}{n} - n^{\theta_n}, \qquad & \lambda_{n_+} &= \frac{j_n}{n} + n^{\theta_n}, \\ \lambda_{p_-} &= \frac{j_p}{p} - n^{\theta_p}, \qquad & \lambda_{p_+} &= \frac{j_p}{p} + n^{\theta_p}, \end{aligned} \tag{2.4}$$

where $\theta_n = (\gamma_n - 1)/2$ and $\theta_p = (\gamma_p - 1)/2$. The Riemann invariants are

$$\begin{aligned} w_n &= \frac{j_n}{n} + \frac{n^{\theta_n}}{\theta_n}, \qquad & z_n &= \frac{j_n}{n} - \frac{n^{\theta_n}}{\theta_n}, \\ w_p &= \frac{j_p}{p} + \frac{p^{\theta_p}}{\theta_p}, \qquad & z_p &= \frac{j_p}{p} - \frac{p^{\theta_p}}{\theta_p}. \end{aligned} \tag{2.5}$$

Therefore, along smooth solutions, the system (BE)-(P) can be written in the diagonal form

$$
\begin{aligned}
\partial_t w_n + \lambda_{n_+} \partial_x w_n &= E - \frac{w_n + z_n}{2\tau} \\
\partial_t z_n + \lambda_{n_-} \partial_x z_n &= E - \frac{w_n + z_n}{2\tau} \\
\partial_t w_p + \lambda_{p_+} \partial_x w_p &= -E - \frac{w_p + z_p}{2\tau} \qquad (2.6)\\
\partial_t z_p + \lambda_{p_-} \partial_x z_p &= -E - \frac{w_p + z_p}{2\tau}
\end{aligned}
$$

$$
\partial_x E = \left(\frac{\theta_n}{2}(w_n - z_n) \right)^{1/\theta_n} - \left(\frac{\theta_p}{2}(w_p - z_p) \right)^{1/\theta_p} - b(x).
$$

We next define, following [22], our finite difference approximating schemes which combines the classical Lax–Friedrichs scheme [17] with an appropriate modification of the fractional step method. In this way it is possible to obtain uniform bounds on the solutions with respect to τ.

To give the approximation solutions $V^l = (n^l, j_n^l, p^l, j_p^l)$, we consider a partition of $\mathbb{R} \times [0, +\infty)$ into horizontal layers

$$
S_k = \{(x,t) | kh \le t < (k+1)h\},
$$

for the time mesh-length $\Delta t = h > 0$ and for any integer $k \ge 0$. Let $I_k = \{i | k + i \text{ is even}\}$ and set

$$
Q_{i,k} = \{(x,t) | (i-1)l < x < (i+1)l\} \cap S_k,
$$

for any $i \in I_k$ and for the space mesh-length $\Delta x = l$, such that the ratio h/l satisfies the CFL condition, namely,

$$
\sup\left(\sup \left| \lambda_{n_\pm}(n^l, j_n^l) \right|, \sup \left| \lambda_{p\pm}(p^l, j_p^l) \right| \right) \le l/2h.
$$

The approximate solution V^l and S_k will be given by means of the solution on $t = kh^-$, for every $k \ge 0$.

Let $V_0 = (n^0, p^0, j_n^0, j_p^0)$ be the vector of the initial data. Set, for any $i \in I_0$,

$$
V^{i,0} = \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} V_0(x)\, dx \qquad (2.7)
$$

and for any $k > 0$ and $i \in I_k$,

$$
V^{i,k} = \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} V^l(x, kh - 0)\, dx. \qquad (2.8)
$$

Therefore, for any $k \geq 0$ and $i \in I_k$, define

$$V_k^l(x) = \begin{cases} V^{i-1,k}, & (i-1)l < x < il, \\ V^{i+1,k}, & il < x < (i+1)l. \end{cases}$$

Let $\tilde{V}^l = (\overline{n}^l, \tilde{j}_n^l, \tilde{p}^l, \tilde{j}_p^l)$ be the solution of Riemann problem for the following homogeneous counterpart of the system (BE) in $Q_{i,k}$, with V_k^l as initial datum at $t = kh$:

$$\begin{aligned} \partial_t n + \partial_x j_n &= 0 \\ \partial_t p + \partial_x j_p &= 0, \\ \partial_t j_n + \partial_x \left(\frac{j_n^2}{n} + r_n \right) &= 0, \\ \partial_t j_p + \partial_x \left(\frac{j_p^2}{p} + r_p \right) &= 0. \end{aligned}$$

This system is formed by two uncoupled homogeneous isentropic gas dynamics systems. The structure of their solutions is well known from [8]. Set, also,

$$\tilde{E}^l(x,t) = \int_{-\infty}^{x} \left[\tilde{n}^l(y,t) - \tilde{p}^l(y,t) - b(y) \right] dy + E^-,$$

where E^- is given by (2.1). Then, we set in S_k

$$\begin{aligned} n^l(x,t) &= \tilde{n}^l(x,t), \\ p^l(x,t) &= \tilde{p}^l(x,t), \\ j_n^l(x,t) &= \left[\tilde{n}^l(x,t) \int_{kh}^{t} e^{-(t-s)/\tau} \tilde{E}^l(x,s)\, ds + e^{-(t-kh)/\tau} \tilde{j}_n^l(x,t) \right], \\ j_p^l(x,t) &= \left[-\tilde{p}^l(x,t) \int_{kh}^{t} e^{-(t-s)/\tau} \tilde{E}^l(x,s)\, ds + e^{-(t-kh)/\tau} \tilde{j}_p^l(x,t) \right]. \end{aligned} \tag{2.9}$$

The approximate electric field E^l is now given by

$$E^l(x,t) = \int_{-\infty}^{x} \left[n^l(y,t) - p^l(y,t) - b(y) \right] dy + E^-. \tag{2.10}$$

Notice that in the same way it is possible to deal with a modified version of the fractional step Godunouv scheme; see [9, 21, 22].

3. EXISTENCE OF UNIFORMLY BOUNDED ENTROPY SOLUTIONS

The main result in this section is the following.

THEOREM 3.1. *Assume that n^0, p^0, j_n^0, j_p^0 are initial data with compact support which satisfy, for some positive constants C_1, C_2,*

$$\begin{aligned} &0 \le n^0(x) \le C_1, \qquad |u_n^0(x)| = \left|\frac{j_n^0(x)}{n^0(x)}\right| \le C_2, \\ &0 \le p^0(x) \le C_1, \qquad |u_p^0(x)| = \left|\frac{j_p^0(x)}{p^0(x)}\right| \le C_2, \end{aligned} \tag{3.1}$$

a.e. $x \in \mathbb{R}$, and assume that $b \in L^1$. Let E^- be some given constant and set

$$E^0(x) = \int_{-\infty}^{x} \bigl(n^0(y) - p^0(y) - b(y)\bigr)\, dy + E^-. \tag{3.2}$$

Then there exists a globally bounded entropy solution to the Cauchy problem (BE)-(P)-(2.1) *and the L^∞-bounds do not depend on τ.*

First we establish some sharp uniform bounds on the approximate solutions $(n^l, p^l, j_n^l, j_p^l, E^l)$ given by the scheme (2.9)–(2.10).

PROPOSITION 3.2. *Under the assumption of the Theorem* 3.1, *there exist some constants C_0, $\overline{E} > 0$ and $l^* > 0$, such that*

$$\begin{aligned} &0 \le n^l(x,t) \le \left[\theta_n\bigl(C_0 + \tau\overline{E}\bigr)\right]^{1/\theta_n}, \\ &0 \le p^l(x,t) \le \left[\theta_p\bigl(C_0 + \tau\overline{E}\bigr)\right]^{1/\theta_p}, \\ &|u_n^l(x,t)| = \left|\frac{j_n^l(x,t)}{n^l(x,t)}\right| \le C_0 + \tau\overline{E}, \\ &|u_p^l(x,t)| = \left|\frac{j_p^l(x,t)}{p^l(x,t)}\right| \le C_0 + \tau\overline{E}, \\ &|E^l(x,t)| \le \overline{E}, \end{aligned} \tag{3.3}$$

for a.e. $(x,t) \in \mathbb{R} \times [0,\infty)$ and any $l \in (0, l^*)$.

Proof. From the definition of the scheme (2.9)–(2.10) we have

$$n^l(x,t) \ge 0, \qquad p^l(x,t) \ge 0 \qquad \text{in } \mathbb{R} \times [0,\infty). \tag{3.4}$$

In fact, the regions $\{\tilde{n}^l \ge 0\}$ and $\{\tilde{p}^l \ge 0\}$ are both invariant for the Riemann problem for homogeneous isentropic gas-dynamics. Also, since

n^0 and p^0 have a compact support, n^l and p^l have still compact support. Next, according to the first and the second equations in (BE), namely, the conservation of densities of electrons and holes, it follows that

$$\int_{-\infty}^{x} n^l(y,t) + p^l(y,t)\,dy \le \|n_0^l\|_{L^1(\mathbb{R})} + \|p_0^l\|_{L^1(\mathbb{R})}. \tag{3.5}$$

Then the supremum norm bound for the electric field follows

$$|E^l(x,t)| \le \|n^0\|_{L^1(\mathbb{R})} + \|p^0\|_{L^1(\mathbb{R})} + \|b\|_{L^1(\mathbb{R})} + E^- =: \overline{E} < +\infty. \tag{3.6}$$

Consider now the discrete Riemann invariants $(w_n^l, z_n^l, w_p^l, z_p^l)$, respectively $(\tilde{w}_n^l, \tilde{z}_n^l, \tilde{w}_p^l, \tilde{z}_p^l)$, given from the relation (2.5) applied to (n^l, p^l, j_n^l, j_p^l), respectively $(\tilde{n}^l, \tilde{p}^l, \tilde{j}_n^l, \tilde{j}_p^l)$. It is possible to show that there exist some constants $K_0 > 0$, $l^* > 0$ such that

$$\begin{aligned} -K_0 &\le z_n^l(x,t) \le w_n^l(x,t) \le K_0, \\ -K_0 &\le z_p^l(x,t) \le w_p^l(x,t) \le K_0, \end{aligned} \tag{3.7}$$

for all $l \in (0, l^*)$ and a.e. $(x,t) \in \mathbb{R} \times [0,\infty)$.

We shall express first $(w_n^l, z_n^l, w_p^l, z_p^l)$ in terms of $(\tilde{w}_n^l, \tilde{z}_n^l, \tilde{w}_p^l, \tilde{z}_p^l)$, by using the scheme (2.9)–(2.10), which reads now, for any $k \ge 0$ and for a.e. $(x,t) \in S_k$,

$$\begin{aligned}
w_n^l(x,t) &= \left(\frac{1+e^{(t-kh)/\tau}}{2}\right)\tilde{w}_n^l(x,t) - \left(\frac{1-e^{-(t-kh)/\tau}}{2}\right)\tilde{z}_n^l(x,t) \\
&\quad + \int_{kh}^{t} e^{-(t-s)/\tau}\tilde{E}^l(x,s)\,ds; \\
z_n^l(x,t) &= \left(\frac{1+e^{-(t-kh)/\tau}}{2}\right)\tilde{z}_n^l(x,t) - \left(\frac{1-e^{-(t-kh)/\tau}}{2}\right)\tilde{w}_n^l(x,t) \\
&\quad + \int_{kh}^{t} e^{-(t-s)/\tau}\tilde{E}^l(x,s)\,ds; \\
w_p^l(x,t) &= \left(\frac{1+e^{-(t-kh)/\tau}}{2}\right)\tilde{w}_p^l(x,t) - \left(\frac{1-e^{-(t-kh)/\tau}}{2}\right)\tilde{z}_p^l(x,t) \\
&\quad - \int_{kh}^{t} e^{-(t-s)/\tau}\tilde{E}^l(x,s)\,ds; \\
z_p^l(x,t) &= \left(\frac{1+e^{-(t-kh)/\tau}}{2}\right)\tilde{z}_p^l(x,t) - \left(\frac{1-e^{-(t-kh)/\tau}}{2}\right)\tilde{w}_p^l(x,t) \\
&\quad - \int_{kh}^{t} e^{-(t-s)/\tau}\tilde{E}^l(x,s)\,ds.
\end{aligned}$$

Set now

$$L_k^l := \max\Big(\sup w_n^l(x, kh+0), -\inf_x z_n^l(x, kh+0),$$

$$\sup_x w_p^l(x, kh+0), -\inf_x z_p^l(x, kh+0)\Big). \tag{3.8}$$

Therefore, by induction, we have

$$\begin{aligned} L_{k+1}^l &\le L_k^l + \bar{E}\int_{kh}^{(k+1)h} e^{-((k+1)h-s)/\tau}\, ds \\ &\le L_0^l + \bar{E}\int_0^{(k+1)h} e^{-((k+1)h-s)/\tau}\, ds \\ &= L_0^l + \tau\bar{E}\left(1 - e^{-(k+1)h/\tau}\right). \end{aligned} \tag{3.9}$$

Then (3.7) follows by setting $K_0 = L_0^l + \tau\bar{E}$. This completes the proof. ∎

Now it is possible, by arguing as in [21], to state a result of convergence for the sequence of the approximate solutions. The proof follows by using the compensated compactness methods [27, 10]. In the case of the nonhomogeneous isentropic gas-dynamics this study was accomplished in [9]. As in well known the main step is in proving that the sequence

$$\Lambda^l := \partial_t \eta(V^l) + \partial_x q(V^l)$$

is relatively compact in H_{loc}^{-1} for any entropy−entropy flux pair (η, q). This follows by using the energy estimates on the approximate state variables, which can obtained as in [21].

PROPOSITION 3.3. *Under the assumptions of Theorem* 3.1 *there is a convergent subsequence* $(n^{l\nu}, p^{l\nu}, j_n^{l\nu}, j_p^{l\nu}, E^{l\nu})$ *and some bounded measurable functions* (n, p, j_n, j_p, E), *such that*

$$\left(n^{l\nu}, p^{l\nu}, j_n^{l\nu}, j_p^{l\nu}\right) \to \left(n, p, j_n, j_p\right)$$

for a.e. $(x, t) \in \mathbb{R} \times [0, \infty)$, *and*

$$E^{l\nu} \to E^l$$

uniformly on the compact subsets of $\mathbb{R} \times [0, \infty)$.

To complete the proof of the Theorem 3.1 we next show the consistency of the approximate sequence $\{V^l, E^l\}$.

PROPOSITION 3.4. *The limit functions* (n, p, j_n, j_p, E), *given by the Proposition* 3.3, *are a bounded entropy solution of the problem* (BE)-(P)-(2.1).

Proof. Set, for a.e. $(x, t) \in \mathbb{R} \times [0, \infty)$,

$$\Gamma_n^l(x, t) := j_n^l - \tilde{j}_n^l, \qquad \Gamma_p^l(x, t) := j_p^l - \tilde{j}_p^l.$$

Thus, for any smooth function $\phi \in C_0^\infty(\mathbb{R} \times (0,\infty))$, we have

$$\iint \left(n^l \phi_t + j_n^l \phi_x \right) dx\, dt$$

$$= \iint \left(\tilde{n}^l \phi_t + \tilde{j}_n^l \phi_x \right) dx\, dt + \iint \Gamma_n^l \phi_n \, dx\, dt =: A_n^l + B_n^l,$$

$$\iint \left(p^l \phi_t + j_p^l \phi_x \right) dx\, dt$$

$$= \iint \left(\tilde{p}^l \phi_t + \tilde{j}_p^l \phi_x \right) dx\, dt + \iint \Gamma_p^l \phi_x dx\, dt =: A_p^l + B_p^l,$$

$$\iint \left[j_n^l \phi_t + \left(\frac{\left(j_n^l\right)^2}{n^l} + r_n(n^l) \right) \phi_x + \left(n^l E^l - \frac{j_n^l}{\tau} \right) \phi \right] dx\, dt$$

$$= \iint \left[\tilde{j}_n^l \phi_t + \left(\frac{\left(\tilde{j}_n^l\right)^2}{\tilde{n}^l} + r_n(\tilde{n}^l) \right) \phi_x + \left(\tilde{n}^l \tilde{E}^l - \frac{\tilde{j}_n^l}{\tau} \right) \phi \right.$$

$$\left. + \iint \left[\Gamma_n^l \phi_t + \left(2 \frac{\tilde{j}_n^l}{\tilde{n}^l} \Gamma_n^l + \frac{\left(\Gamma_n^l\right)^2}{\tilde{n}^l} \right) \phi_x - \frac{\Gamma_n^l}{\tau} \phi \right] dx\, dt \right.$$

$$=: C_n^l + D_n^l,$$

$$\iint \left[j_p^l \phi_t + \left(\frac{\left(j_p^l\right)^2}{p^l} + r_p(p^l) \right) \phi_x + \left(-p^l E^l - \frac{j_p^l}{\tau} \right) \phi \right] dx\, dt$$

$$= \iint \left[\tilde{j}_p^l \phi_t + \left(\frac{\left(\tilde{j}_p^l\right)^2}{\tilde{p}^l} + r_p(\tilde{p}^l) \right) \phi_x + \left(-\tilde{p}^l \tilde{E}^l - \frac{\tilde{j}_p^l}{\tau} \right) \phi \right.$$

$$\left. + \iint \left[\Gamma_p^l \phi_t + \left(2 \frac{\tilde{j}_p^l}{\tilde{p}^l} \Gamma_p^l + \frac{\left(\Gamma_p^l\right)^2}{\tilde{p}^l} \right) \phi_x - \frac{\Gamma_p^l}{\tau} \phi \right] dx\, dt \right.$$

$$=: C_p^l + D_p^l.$$

Now observe that, from (2.9) and the estimates (3.3), we have

$$|\Gamma_n^l(x,t)|, \qquad |\Gamma_p^l(x,t)| \le Ch.$$

Then for some constant $\tilde{C} > 0$,

$$|B_n^l| + |B_p^l| \le h \iint 2C |\phi_x| \, dx\, dt \le \tilde{C} h$$

and

$$|D_n^l| + |D_p^l| \le h \iint \left\{ C|\phi_t| + \left[2C\left|\frac{\tilde{j}_n^l}{\tilde{n}^l}\right| + \left|\frac{\tilde{j}_p^l}{\tilde{p}^l}\right| + hC^2\left(\frac{1}{\tilde{n}^l} + \frac{1}{\tilde{p}^l}\right)\right]|\phi_x| + \frac{2C}{\tau}|\phi| \right\} dx\,dt \le \tilde{C}h;$$

hence, $B_n^l, B_p^l, D_n^l, D_p^l \to 0$ as $l \to 0^+$.

The term A_n^l, A_p^l are estimated as in [21] and they converge to zero as $l \to 0^+$.

Consider now

$$\begin{aligned} C_n^l &= \sum_k \int \left(\tilde{j}_n^l(x, kh+0) - \tilde{j}_n^l(x, kh-0) \right) \phi(x, kh)\, dx \\ &\quad + \iint \left(\tilde{n}^l \tilde{E}^l - \frac{\tilde{j}_n^l}{\tau} \right) \phi\, dx\, dt \\ &=: C_{n1}^l + C_{n2}^l, \\ C_p^l &= \sum_k \int \left(\tilde{j}_p^l(x, kh+0) - \tilde{j}_p^l(x, kh-0) \right) \phi(x, kh)\, dx \\ &\quad + \iint \left(-\tilde{p}^l \tilde{E}^l - \frac{\tilde{j}_p^l}{\tau} \right) \phi\, dx\, dt \\ &=: C_{p1}^l + C_{p2}^l. \end{aligned}$$

Set $\phi^{ik} := \phi(il, kh)$. We have

$$\begin{aligned} |C_{n1}^l| &= \left| \sum_{i,k} \int_{(i-1)l}^{(i+1)l} \left(\tilde{j}_n^l(x, kh+0) - \tilde{j}_n^l(x, kh-0) \right)\left(\phi(x, kh) - \phi^{ik} \right) dx \right| \\ &\le C l^{1/2} \|\phi\|_{C_0^1} \left\{ \sum_{\substack{i,k \\ |ih| \le L}} \int_{(i-1)l}^{(i+1)l} |\tilde{j}_n^l(x, kh+0) - \tilde{j}_n^l(x, kh-0)|^2\, dx \right\}^{1/2} \\ &\le C l^{1/2}. \end{aligned}$$

Here the energy estimates of [21, Theorem 5.1] on the approximation state variables have been used. The term C_{p1}^l is estimated in the same way. Thus $C_{n1}^l, C_{p1}^l \to 0$ as $l \to 0^+$.

The other terms are

$$C_{n2}^l = \sum_{i,k} \phi^{ik} \int_{(i-1)l}^{(i+1)l} \left[\tilde{j}_n^l(x, kh-0) - \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} \tilde{j}_n^l(s, kh-0)\, ds \right.$$
$$\left. - \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} \Gamma_n^l(s, kh-0)\, ds \right] dx + \iint \left(\tilde{n}^l \tilde{E}^l - \frac{\tilde{j}_n^l}{\tau} \right) \phi\, dx\, dt,$$

$$C_{p2}^l = \sum_{i,k} \phi^{ik} \int_{(i-1)l}^{(i+1)l} \left[\tilde{j}_p^l(x, kh-0) - \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} \tilde{j}_p^l(s, kh-0)\, ds \right.$$
$$\left. - \frac{1}{2l} \int_{(i-1)l}^{(i+1)l} \Gamma_p^l(s, kh-0)\, ds \right] dx + \iint \left(-\tilde{p}^l \tilde{E}^l - \frac{\tilde{j}_p^l}{\tau} \right) \phi\, dx\, dt.$$

They are estimated following [21, 22, 9]. So we proved that the limit functions (n, p, j_n, j_p, E), given by the Proposition 3.3, are a bounded weak solution of the problem (BE)-(P)-(2.1). To prove the entropy inequality (2.3) take now any convex entropy function $\eta = \eta(V^l)$ and let $q = q(V^l)$ be the correspondent entropy-flux. Recalling that $G(V, E) = (0, 0, nE - j_n/\tau, -pE - j_p/\tau)$, for any nonnegative function $\phi \in C_0^\infty(\mathbb{R} \times (0, \infty))$ we have

$$I^l = \iint \eta(V^l)\phi_t + q(V^l)\phi_x + \left(\nabla\eta(V^l)\right)^{\mathrm{T}} \cdot G(V^l, E^l)\phi\, dx\, dt$$
$$= \iint \eta(\tilde{V}^l)\phi_t + q(\tilde{V}^l)\phi_x + \left(\nabla\eta(\tilde{V}^l)\right)^{\mathrm{T}} \cdot G(\tilde{V}^l, \tilde{E}^l)\phi\, dx\, dt$$
$$+ \sum_{k=1}^{N} \iint_{S_{k-1}} \Big\{ \left[\eta(V^l) - \eta(\tilde{V}^l)\right]\phi_t + \left[q(V^l) - q(\tilde{V}^l)\right]\phi_x$$
$$+ \left[\left(\nabla\eta(V^l)\right)^{\mathrm{T}} \cdot G(V^l, E^l) - \left(\nabla\eta(\tilde{V}^l)\right)^{\mathrm{T}} \cdot G(\tilde{V}^l, \tilde{E}^l) \right]\phi \Big\}\, dx\, dt$$
$$= F_1^l + F_2^l.$$

Since the functions $\eta, q, \nabla\eta, G$ are smooth, from (2.9) and (3.3) there exists a constant C such that

$$|F_2^l| \leq Cl.$$

Finally the term F_1^l is estimated as the terms C_n^l, C_p^l, by using, moreover, the convexity of η, to obtain

$$I^l \geq -C\sqrt{h}$$

and in the limit the inequality (2.3). This completes the proof. ∎

4. THE DRIFT–DIFFUSION MODEL

In this section we shall consider the drift–diffusion model (DD)-(P) for a bipolar semiconductor with initial conditions

$$\begin{aligned} &\mathscr{N}(x,0) = \mathscr{N}^0(x), \qquad \mathscr{P}(x,0) = \mathscr{P}^0(x) \\ &\mathscr{E}(x,0) = \mathscr{E}^0(x) = \int_{-\infty}^{x} \left(\mathscr{N}^0(y) - \mathscr{P}^0(y) - b(y)\right) dy + \mathscr{E}^-, \end{aligned} \tag{4.1}$$

where $\mathscr{N}^0$, P^0 are some given nonnegative bounded measurable functions with compact support and $\mathscr{E}^-$ is a constant.

The locally bounded measurable functions $(\mathscr{N}, \mathscr{P}, \mathscr{E})$ are a weak solution to the Cauchy problem for (DD)-(P)-(4.1) in the strip $\mathbb{R} \times (0, T)$, for $T > 0$, if $\partial_x(r_n(\mathscr{N}))$ and $\partial_x(r_p(\mathscr{P}))$ are measurable bounded functions and, for any smooth test function ϕ with compact support in $\mathbb{R} \times [0, T)$, one has

$$\iint \left(\mathscr{N}\partial_t \phi - \left(\partial_x r_n(\mathscr{N}) - \mathscr{N}\mathscr{E}\right)\partial_x \phi\right) dx\, dt + \int_{\{t=0\}} \mathscr{N}^0 \phi\, dx = 0, \tag{4.2}$$

$$\iint \left(\mathscr{P}\partial_t \phi - \left(\partial_x r_p(\mathscr{P}) + \mathscr{P}\mathscr{E}\right)\partial_x \phi\right) dx\, dt + \int_{\{t=0\}} \mathscr{P}^0 \phi\, dx = 0, \tag{4.3}$$

$$\iint \{\mathscr{E}\partial_x \phi + (\mathscr{N} - \mathscr{P} - b)\phi\}\, dx\, dt = 0. \tag{4.4}$$

We require also that there exists a constant $\mathscr{E}^-$:

$$\lim_{x \to -\infty} \mathscr{E}(x,t) = \mathscr{E}^-, \tag{4.5}$$

for all $t \geq 0$.

THEOREM 4.1. *Assume that $\mathscr{N}^0$, $\mathscr{P}^0$ are bounded nonnegative initial data with compact support. Then for any $T > 0$ there exists a weak solution $(\mathscr{N}, \mathscr{P}, \mathscr{E})$ of problem (DD)-(P)-(4.1) in the strip $\mathbb{R} \times (0, T)$. Moreover, the electron and hole current densities are given by $\mathscr{J}_n = \mathscr{N}\mathscr{E} - \partial_x r_n(\mathscr{N})$, $\mathscr{J}_p = -\mathscr{P}\mathscr{E} - \partial_x r_p(\mathscr{P})$ and belong to L^2.*

We establish the existence of these solutions as the limit of a scaled sequence of solutions to the Cauchy problem (BE)-(P) when the relaxation time τ, given by (1.1), goes to zero.

For any fixed $\tau > 0$ we consider the entropy solutions $(n^\tau, j_n^\tau, p^\tau, j_p^\tau, \mathscr{E}^\tau)$ to the Cauchy problem (BE)-(P) in $\mathbb{R} \times [0, \infty)$, with initial data

$(\mathcal{N}^0, \mathcal{P}^0, j_n^0, j_p^0)$. We introduce the scaled variables

$$\mathcal{N}^\tau(x,t) := n^\tau\left(x, \frac{t}{\tau}\right), \qquad \mathcal{P}^\tau(x,t) := p^\tau\left(x, \frac{t}{\tau}\right),$$

$$\mathcal{J}_n{}^\tau(x,t) := \frac{1}{\tau} j_n^\tau\left(x, \frac{t}{\tau}\right), \qquad \mathcal{J}_p{}^\tau(x,t) := \frac{1}{\tau} j_p^\tau\left(x, \frac{t}{\tau}\right),$$

$$\mathcal{E}^\tau(x,t) := E^\tau\left(x, \frac{t}{\tau}\right).$$

These new variables satisfy the following relaxing system:

$$\begin{aligned}
&\partial_t \mathcal{N}^\tau + \partial_x \mathcal{J}_n{}^\tau = 0,\\
&\partial_t \mathcal{P}^\tau + \partial_x \mathcal{J}_p{}^\tau = 0,\\
&\partial_t\left(\tau^2 \mathcal{J}_n{}^\tau\right) + \partial_x\left(\tau^2 \frac{\mathcal{J}_n{}^{\tau 2}}{\mathcal{N}^\tau} + r_n(\mathcal{N}^\tau)\right) = \mathcal{N}^\tau \mathcal{E}^\tau - \mathcal{J}_n{}^\tau,\\
&\partial_t\left(\tau^2 \mathcal{J}_p{}^\tau\right) + \partial_x\left(\tau^2 \frac{\mathcal{J}_p{}^{\tau 2}}{\mathcal{P}^\tau} + r_p(\mathcal{P}^\tau)\right) = -\mathcal{P}^\tau \mathcal{E}^\tau - \mathcal{J}_p{}^\tau,\\
&\partial_x \mathcal{E}^\tau = \mathcal{N}^\tau - \mathcal{P}^\tau - b(x).
\end{aligned} \tag{RS$_\tau$}$$

Theorem 4.1 follows easily from the next result.

THEOREM 4.2. *Let $\{\mathcal{N}^\tau, \mathcal{P}^\tau, \mathcal{J}_n{}^\tau, \mathcal{J}_p{}^\tau, \mathcal{E}^\tau\}$ be the sequence of solutions to* $(\mathrm{RS})_\tau$ *given by Theorem* 3.1 *with initial data* $(\mathcal{N}^0, \mathcal{P}^0, (1/\tau)j_{n^\tau}^0, (1/\tau)j_{p^\tau}^0)$. *Then there exist $\mathcal{N}, \mathcal{P} \in L^\infty$ such that $\{\mathcal{N}^\tau, \tau\mathcal{J}_n{}^\tau, \mathcal{P}^\tau, \tau\mathcal{J}_p{}^\tau\}$ converges in L^p_{loc} strongly to $\{\mathcal{N}, 0, \mathcal{P}, 0\}$, for all $p \in (1, +\infty]$, and there exists a Lipschitz continuous function $\mathcal{E}$ such that $\{\mathcal{E}^\tau\}$ converges uniformly on compacta to $\mathcal{E}$ as $\tau \to 0^+$. The limit densities $(\mathcal{N}, \mathcal{P}, \mathcal{E})$ are weak solutions of the Cauchy problem for the drift–diffusion model* (DD)-(P). *Moreover, let $\mathcal{J}_n = w - \lim \mathcal{J}_n{}^\tau$, $\mathcal{J}_p = w - \lim \mathcal{J}_p{}^\tau$ in L^2, as $\tau \downarrow 0$, it follows that*

$$\mathcal{J}_n = \mathcal{N}\mathcal{E} - \partial_x r_n(\mathcal{N}), \qquad \mathcal{J}_p = -\mathcal{P}\mathcal{E} - \partial_x r_p(\mathcal{P}). \tag{4.6}$$

In order to prove Theorems 4.1 and 4.2 we need some preliminary results. The main ingredient is still the theory of compensated compactness which provides the following lemma [27].

LEMMA 4.3. *Given two sequences $\{U^\varepsilon\}$ and $\{V^\varepsilon\}$, uniformly bounded in L^2_{loc} assume that $\{\operatorname{div} U^\varepsilon\}$ and $\{\operatorname{curl} V^\varepsilon\}$ belong to a bounded set of L^2_{loc} independent of ε: then $U^\varepsilon \cdot V^\varepsilon \rightharpoonup U \cdot V$ in $\mathcal{D}'$, where $U = w - \lim U^\varepsilon$ and $V = w - \lim V^\varepsilon$.*

To apply this result we have to use the L^∞ estimates given in Section 3 and the following energy estimates.

PROPOSITION 4.4. *The entropy solutions to* $(\mathrm{RS})_\tau$ *verify the inequality*

$$\int_0^T \int \frac{\mathcal{J}_n^{\tau^2}(x,t)}{\mathcal{N}^\tau(x,t)} + \frac{\mathcal{J}_p^{\tau^2}(x,t)}{\mathcal{P}^\tau(x,t)}\,dx\,dt \le C_T, \tag{4.7}$$

for all $T > 0$, *where the constant* C_T *depends only on* T.

Proof. Consider the mechanical entropy associated to the system (BE),

$$\eta^* = \frac{j_n^{\tau^2}}{2n^\tau} + \frac{j_p^{\tau^2}}{2p^\tau} + \frac{(n^\tau)^{\gamma_n}}{\gamma_n(\gamma_n - 1)} + \frac{(p^\tau)^{\gamma_p}}{\gamma_p(\gamma_p - 1)}.$$

From the entropy inequality we have, for a.e. $t \ge 0$:

$$\partial_t \int \eta^*(x,t)\,dx \le \int\left((j_n^\tau - j_p^\tau)E^\tau - \frac{1}{\tau}\left(\frac{j_n^{\tau^2}}{n^\tau} + \frac{j_p^{\tau^2}}{p^\tau}\right)\right)(x,t)\,dx. \tag{4.8}$$

Set

$$f(t) := \int \eta^*(x,t)\,dx$$

and

$$\Psi(t) := \int\left(\frac{j_n^{\tau^2}}{n^\tau} + \frac{j_p^{\tau^2}}{p^\tau}\right)(x,t)\,dx.$$

Now, from the boundedness of n^τ and p^τ in L^1 and E^τ in L^∞ we obtain

$$\begin{aligned}\left|\int j_n^\tau E^\tau\,dx\right| &\le \left(\int n^\tau E^{\tau^2}\,dx\right)^{1/2}\left(\int \frac{j_n^{\tau^2}}{n^\tau}\,dx\right)^{1/2}\\ &\le \|\mathcal{N}^0\|_1^{1/2}\overline{E}\sqrt{\Psi}\end{aligned}$$

and

$$\begin{aligned}\left|\int j_p^\tau E^\tau\,dx\right| &\le \left(\int p^\tau E^{\tau^2}\,dx\right)^{1/2}\left(\int \frac{j_p^{\tau^2}}{n^\tau}\,dx\right)^{1/2}\\ &\le \|\mathcal{P}^0\|_1^{1/2}\overline{E}\sqrt{\Psi}\,.\end{aligned}$$

Thus, it follows that

$$\frac{d}{dt}f \le C\sqrt{\Psi} - \frac{1}{\tau}\Psi. \tag{4.9}$$

Next, we set

$$F(t) := f(t/\tau)$$

and

$$\Phi(t) := \frac{1}{\tau^2}\Psi\left(\frac{t}{\tau}\right) = \int\left(\frac{\mathscr{J}_n^{\tau^2}(x,t)}{\mathscr{N}^\tau(x,t)} + \frac{\mathscr{J}_p^{\tau^2}(x,t)}{\mathscr{P}^\tau(x,t)}\right)dx,$$

by considering the scaled variables $\mathscr{N}^\tau, \mathscr{J}_n^\tau, \mathscr{P}^\tau, \mathscr{J}_p^\tau$. Hence, from (4.9), we have

$$\frac{d}{dt}F \leq C\sqrt{\Phi} - \Phi.$$

Therefore, for all $t \geq 0$,

$$F(t) - F(0) \leq C\int_0^t \sqrt{\Phi(s)}\,ds - \int_0^t \Phi(s)\,ds.$$

Set

$$Y(t) := \frac{1}{t}\int_0^t \Phi(s)\,ds.$$

Then, from the Hölder inequality, we have

$$Y(t) + \frac{F(t)}{t} \leq \frac{F(0)}{t} + C\sqrt{Y(t)}\,.$$

Hence, since $F(t) \geq 0$, it is easy to show that there exist a constant $\tilde{C} > 0$ and $t^0 > 0$ such that, for all $t \geq t^0$, we have

$$Y(t) \leq \tilde{C}.$$

Now, the inequality (4.7) follows easily. ∎

Next, we have the following.

PROPOSITION 4.5. *The sequences* $\{\mathscr{N}^\tau, \mathscr{P}^\tau\}$, $\{\tau\mathscr{J}_n^\tau, \tau\mathscr{J}_p^\tau\}$, *and* $\{(\mathscr{J}_n^\tau)^2/\mathscr{N}^\tau, (\mathscr{J}_p^\tau)^2/\mathscr{P}^\tau\}$ *are uniformly bounded in* $L^\infty(\mathbb{R}\times(0,\infty))$, $L^\infty(\mathbb{R}\times(0,\infty))$, *and* $L^1(\mathbb{R}\times(0,T))$, *respectively, for all* $T > 0$. *Therefore, if* $\mathscr{N} := w^* - \lim \mathscr{N}^\tau$, $\mathscr{P} := w^* - \lim \mathscr{P}^\tau$, $\tilde{r}_n = w^* - \lim r_n(\mathscr{N}^\tau)$, $\tilde{r}_p = w^* - \lim r_p(\mathscr{P}^\tau)$ *as* $\tau \to 0^+$, *we have*

$$\tilde{r}_n = r_n(\mathscr{N}), \qquad \tilde{r}_p = r_p(\mathscr{P}), \tag{4.10}$$

$$\begin{aligned} \mathscr{N}^\tau r_n(\mathscr{N}^\tau) &\rightharpoonup \mathscr{N} r_n(\mathscr{N}) \qquad \text{in } \mathscr{D}', \\ \mathscr{P}^\tau r_p(\mathscr{P}^\tau) &\rightharpoonup \mathscr{P} r_p(\mathscr{P}) \qquad \text{in } \mathscr{D}', \end{aligned} \tag{4.11}$$

$$\mathscr{N}^\tau \to \mathscr{N},\ \mathscr{P}^\tau \to \mathscr{P} \qquad \text{in } L^q_{\text{loc}} \text{ strongly for all } q \in [1, +\infty). \tag{4.12}$$

Proof. Set $\mathscr{U}_{\mathscr{N}^\tau} = \{\mathscr{N}^\tau, \mathscr{I}_n^{\ \tau}\}$ and $\mathscr{U}_{\mathscr{P}^\tau} = \{\mathscr{P}^\tau, \mathscr{I}_p^{\ \tau}\}$. The sequence $\{\mathscr{N}^\tau, \mathscr{P}^\tau\}$ is uniformly bounded in L^∞ from Proposition 3.2. On the contrary $\mathscr{I}_n^{\ \tau}$ and $\mathscr{I}_p^{\ \tau}$ may be not uniformly bounded in L^∞, but from Proposition 4.3 we have

$$\|\mathscr{I}_n^{\ \tau}\|_{L^2} \le \left\| \frac{(\mathscr{I}_n^{\ \tau})^2}{\mathscr{N}^\tau} \right\|_{L^1} \|\mathscr{N}^\tau\|_{L^\infty},$$

$$\|\mathscr{I}_p^{\ \tau}\|_{L^2} \le \left\| \frac{(\mathscr{I}_p^{\ \tau})^2}{\mathscr{P}^\tau} \right\|_{L^1} \|\mathscr{P}^\tau\|_{L^\infty}.$$

So $\{\mathscr{U}_{\mathscr{N}^\tau}\}$ and $\{\mathscr{U}_{\mathscr{P}^\tau}\}$ are uniformly bounded in $L^2(\Omega)$ for any relatively compact open region $\Omega \subseteq \mathbb{R} \times (0, \infty)$. Moreover, the first two equations in $(\mathrm{RS})_\tau$ become

$$\operatorname{div} \mathscr{U}_{\mathscr{N}^\tau} = \operatorname{div} \mathscr{U}_{\mathscr{P}^\tau} = 0.$$

Next, setting

$$\mathscr{V}_{\mathscr{N}^\tau} := \left\{ -\tau^2 \frac{(\mathscr{I}_n^{\ \tau})^2}{\mathscr{N}^\tau} - r_n(\mathscr{N}^\tau), \tau^2 \mathscr{I}_n^{\ \tau} \right\},$$

$$\mathscr{V}_{\mathscr{P}^\tau} := \left\{ -\tau^2 \frac{(\mathscr{I}_p^{\ \tau})^2}{\mathscr{P}^\tau} - r_p(\mathscr{P}^\tau), \tau^2 \mathscr{I}_p^{\ \tau} \right\},$$

the second two equations in $(\mathrm{RS})_\tau$ give that

$$\{\operatorname{curl} \mathscr{V}_{\mathscr{N}^\tau}, \operatorname{curl} \mathscr{V}_{\mathscr{P}^\tau}\} \in \left\{\text{a bounded set in } L^2(\Omega)\right\}$$

for any relatively compact open region $\Omega \subseteq \mathbb{R} \times (0, \infty)$; in fact, the sequences $\{\mathscr{N}^\tau \mathscr{E}^\tau\}, \{\mathscr{P}^\tau \mathscr{E}^\tau\}$ are uniformly bounded in $L^2(\mathbb{R} \times [0, T])$, for all $T > 0$, since

$$\int_0^\tau \int |\mathscr{N}^\tau \mathscr{E}^\tau|^2 + |\mathscr{P}^\tau \mathscr{E}^\tau|^2 \, dx \le T\overline{E}(\|\mathscr{N}^\tau\|_{L^1} \|\mathscr{N}^\tau\|_{L^\infty} + \|\mathscr{P}^\tau\|_{L^1} \|\mathscr{P}^\tau\|_{L^\infty}). \tag{4.13}$$

Set

$$\mathscr{U}_{\mathscr{N}} = w - \lim \mathscr{U}_{\mathscr{N}^\tau} =: \{\mathscr{N}, \mathscr{I}_n\}, \qquad \mathscr{U}_{\mathscr{P}} = w - \lim \mathscr{U}_{\mathscr{P}^\tau} =: \{\mathscr{P}, \mathscr{I}_p\}$$

and

$$\mathscr{N}_{\mathscr{N}} = w - \lim \mathscr{V}_{\mathscr{N}^\tau} = \{\chi_{\mathscr{N}}, \xi_{\mathscr{N}}\}, \qquad \mathscr{V}_{\mathscr{P}} = w - \lim \mathscr{V}_{\mathscr{P}^\tau} = \{\chi_{\mathscr{P}}, \xi_{\mathscr{P}}\},$$

where

$$\chi_{\mathcal{N}} = w - \lim\left(-\tau^2 \frac{(\mathcal{J}_n^\tau)^2}{\mathcal{N}^\tau} - r_n(\mathcal{N}^\tau)\right), \qquad \xi_{\mathcal{N}} = w - \lim\left(\tau^2 \mathcal{J}_n^\tau\right),$$

$$\chi_{\mathcal{P}} = w - \lim\left(-\tau^2 \frac{(\mathcal{J}_p^\tau)^2}{\mathcal{P}^\tau} - r_p(\mathcal{P}^\tau)\right), \qquad \xi_{\mathcal{P}} = w - \lim\left(\tau^2 \mathcal{J}_p^\tau\right).$$

Hence, by using Lemma 4.3, we deduce that the products

$$\mathcal{U}_{\mathcal{N}^\tau} \cdot \mathcal{V}_{\mathcal{N}^\tau} = \tau^2 |\mathcal{J}_n^\tau|^2 - \mathcal{N}^\tau r_n(\mathcal{N}^\tau) - \tau^2 |\mathcal{J}_n^\tau|^2 = -\mathcal{N}^\tau r_n(\mathcal{N}^\tau),$$

$$\mathcal{U}_{\mathcal{P}^\tau} \cdot \mathcal{V}_{\mathcal{P}^\tau} = \tau^2 |\mathcal{J}_p^\tau|^2 - \mathcal{P}^\tau r_p(\mathcal{P}^\tau) - \tau^2 |\mathcal{J}_p^\tau|^2 = -\mathcal{P}^\tau r_p(\mathcal{P}^\tau),$$

converge in $\mathcal{D}'$ to $\mathcal{U}_{\mathcal{N}} \cdot \mathcal{V}_{\mathcal{N}}$, respectively, to $\mathcal{U}_{\mathcal{P}} \cdot \mathcal{V}_{\mathcal{P}}$.

In particular, we have that $\xi_{\mathcal{N}} = \xi_{\mathcal{P}} = 0$, since, as $\tau \to 0$, $\tau \mathcal{J}_n^\tau$ and $\tau \mathcal{J}_p^\tau$ tend to zero a.e. in $\mathbb{R} \times (0, \infty)$; furthermore, $\chi_{\mathcal{N}} = w - \lim(-r_n(\mathcal{N}^\tau)) = -\tilde{r}_n$ and $\chi_{\mathcal{P}} = w - \lim(-r_p(\mathcal{P}^\tau)) = -\tilde{r}_p$ and then (4.11) follows.

The equalities in (4.10) follow as in [22] by using Minty's argument and the monotonicity of the functions r_n, r_p. Finally, we use the convexity of r_n, r_p, and the basic properties of the Young measures to show (4.12) (see [20, Proposition 4.3]). ∎

Proof of Theorem 4.2. The strong convergence is a consequence of the Proposition 4.5. The second two equations of $(\mathrm{RS})_\tau$ give

$$\iint \left\{ \tau^2 \zeta_t \mathcal{J}_n^\tau + \tau^2 \zeta_x \frac{(\mathcal{J}_n^\tau)^2}{\mathcal{N}^\tau} + \zeta_x r_n(\mathcal{N}^\tau) \right\} dx\, dt = \iint \{\mathcal{N}^\tau \mathcal{E}^\tau - \mathcal{J}_n^\tau\} \zeta \, dx\, dt,$$

$$\iint \left\{ \tau^2 \zeta_t \mathcal{J}_p^\tau + \tau^2 \zeta_x \frac{(\mathcal{J}_p^\tau)^2}{\mathcal{P}^\tau} + \zeta_x r_p(\mathcal{P}^\tau) \right\} dx\, dt = \iint \{-\mathcal{P}^\tau \mathcal{E}^\tau - \mathcal{J}_p^\tau\} \zeta \, dx\, dt,$$

for all $\zeta \in \mathcal{D}(\mathbb{R} \times (0, \infty))$. The first and the second terms to the left-hand side of the above identities vanish as $\pi \downarrow 0$, so that we obtain

$$\iint r_n(\mathcal{N}) \zeta_x \, dx\, dt = \iint \{\mathcal{N}\mathcal{E} - \mathcal{J}_n\} \zeta \, dx\, dt$$

and

$$\iint r_p(\mathcal{P}) \zeta_x \, dx\, dt = \iint \{-\mathcal{P}\mathcal{E} - \mathcal{J}_p\} \zeta \, dx\, dt,$$

where

$$\mathcal{E}(x,t) = \mathcal{E}^- + \int_{-\infty}^x \{\mathcal{N}(y,t) - \mathcal{P}(y,t) - b(y)\}\, dy.$$

Finally, $\partial_x r_n(\mathcal{N}), \partial_x r_p(\mathcal{P}) \in L^2$ since $\mathcal{N}\mathcal{E} - \mathcal{I}_n, -\mathcal{P}\mathcal{E} - \mathcal{I}_p \in L^2$ and from the continuity equation, we recover (5.2) and (5.3). ∎

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